Discrete Fourier Series: Introduction and Basics.

Note: We will change some notation...

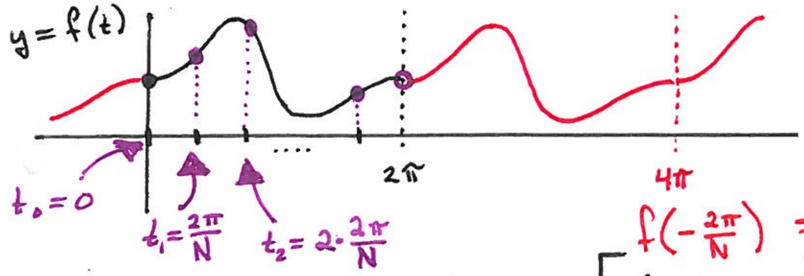
n should be a vector index

(i.e. v_n = "n^{th} element of $\underline{v}$")

So we will use k for Fourier coeff.s

(instead of $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-int}\, dt$

use $c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-ikt}\, dt$)



$f(-\frac{2\pi}{N}) = f((N-1)\frac{2\pi}{N})$

function $f(t)$ $\rightsquigarrow$ vector

$$\underline{f} = \begin{bmatrix} f(0) \\ f(\frac{2\pi}{N}) \\ f(2\cdot\frac{2\pi}{N}) \\ \vdots \\ f((N-1)\frac{2\pi}{N}) \end{bmatrix} \begin{array}{l} \leftarrow f_0 \\ \leftarrow f_1 \\ \leftarrow f_2 \\ \\ \leftarrow f_{N-1} \end{array}$$

$f(N\frac{2\pi}{N}) = f(0)$

$f((N+1)\frac{2\pi}{N}) = f(\frac{2\pi}{N})$

Note: $\underline{f}_n$ is $f(t)$ at $t = n\frac{2\pi}{N}$

Discretize Continuous Fourier Series:

$f(t)$ periodic with period 2π

$$\begin{cases} f(t) = \sum_{-\infty}^{\infty} c_k e^{ikt} \\ c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-ikt}\, dt \end{cases}$$

$$= \frac{1}{2\pi}\int_{0}^{2\pi} f(t)\, e^{-ikt}\, dt$$

since $f(t)$ is periodic.

→ To compute c_k we need to know $f(t)$ for one period, say $0 \le t < 2\pi$

Discretize $f(t)$ on $0 \le t < 2\pi$ to be a length N vector $\underline{f}$

→ Plug into continuous formulas:

$$f(t) = \sum c_k e^{ikt} \qquad c_k = \frac{1}{2\pi}\int_0^{2\pi} f(t)\, e^{-ikt}\, dt$$

$$\underline{f}_n = \sum_{k=0}^{N-1} c_k e^{ik(n\frac{2\pi}{n})} \qquad c_k = \frac{1}{2\pi}\sum_{n=0}^{N-1} \underline{f}_n\, e^{-ik(n\frac{2\pi}{N})}\, \frac{2\pi}{N}$$

$$= \sum_{k=0}^{N-1} c_k e^{(\frac{2\pi i}{N})k\cdot n} \qquad c_k = \frac{1}{N}\sum_{n=0}^{N-1} \underline{f}_n\, e^{(-\frac{2\pi i}{N})k\cdot n}$$

Discrete Fourier — so far:

$\underline{f}$ is a vector of length N

↳ Fourier coefficients:

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} f_n e^{\left(-\frac{2\pi i}{N}\right)\cdot kn}$$

$\mathcal{F}\{\underline{f}\} = (c_0, c_1, \dots, c_{N-1}) = \underline{c}$

is the "Discrete Fourier Transform"

↳ Recover $\underline{f}$ from coefficients

$$f_n = \sum_{k=0}^{N-1} c_k e^{\left(\frac{2\pi i}{N}\right)\cdot nk}$$

$\mathcal{F}^{-1}\{\underline{c}\} = \underline{f}$

is the "Inverse Fourier Transform"

Remark: In "time domain" (i.e. $f(t)$)

we will use vector subscript $\underline{\underline{n}} \rightsquigarrow f_n$

In "frequency domain" (i.e. Fourier coeff)

we will use vector subscript $\underline{\underline{k}} \rightsquigarrow c_k$

→ This is like using "t" for functions $\underline{f(t)}$

and "s" for Laplace transforms $\underline{\mathcal{L}\{f(t)\}}$

Do not fear the $e^{\left(\frac{2\pi i}{N}\right)}$

→ It is just a "Root of Unity"

Write ω_N ("Omega - N") for $e^{\left(\frac{2\pi i}{N}\right)}$

$\omega_N = e^{\left(\frac{2\pi i}{N}\right)}$ is an N^{th} root of unity

— meaning: $(\omega_N)^N = \left(e^{\frac{2\pi i}{N}}\right)^N = e^{2\pi i} = 1.$

Roots of unity:

$$\omega_N = e^{\left(\frac{2\pi i}{N}\right)} = \cos\left(\frac{2\pi}{N}\right) + i\sin\left(\frac{2\pi}{N}\right)$$

Basic examples:

$$\omega_1 = \cos(2\pi) + i\sin(2\pi) = 1$$

$$\omega_2 = \cos\left(\frac{2\pi}{2}\right) + i\sin\left(\frac{2\pi}{2}\right) = -1$$

$$\omega_4 = \cos\left(2\pi/4\right) + i\sin\left(2\pi/4\right) = i$$

$$\omega_8 = \cos\left(2\pi/8\right) + i\sin\left(2\pi/8\right) = \frac{\sqrt{2} + \sqrt{2}i}{2}$$

$$\omega_3 = \cos\left(2\pi/3\right) + i\sin\left(2\pi/3\right) = \frac{-1 + \sqrt{3}i}{2}$$

$$\omega_6 = \cos\left(2\pi/6\right) + i\sin\left(2\pi/6\right) = \frac{1 + \sqrt{3}i}{2}$$

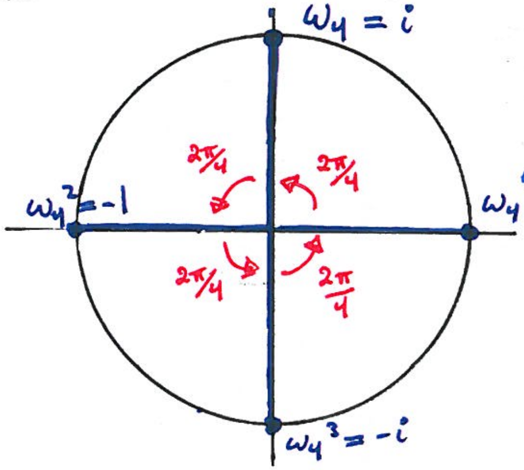
$$\omega_{12} = \cos\left(2\pi/12\right) + i\sin\left(2\pi/12\right) = \frac{\sqrt{3} + i}{2}$$

Easy way to understand ω_N and ω_N^k:

- Consider the circle with radius 1 in $\mathbb{C}$
- ω_N is at the point $1/N$ around the circle
- Powers ω_N^k are given by moving around the circle with steps of size ω_N

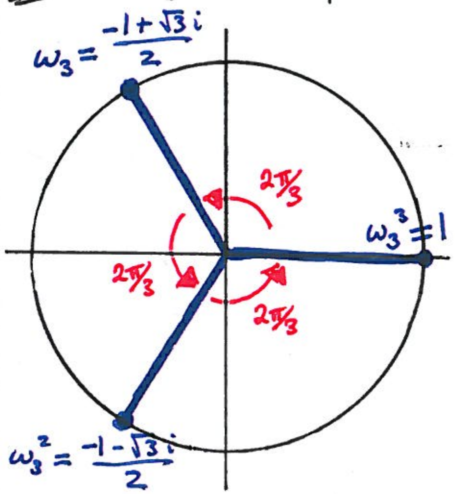
(Coordinates are <u>always</u> cos & sin)

EX: ω_4 and its powers



Divide the circle into 4 pieces. $\omega_N^0 = 1$.

EX: ω_3 and powers



Divide the circle into 3 pieces.

Powers of ω_N are <u>modular</u>:

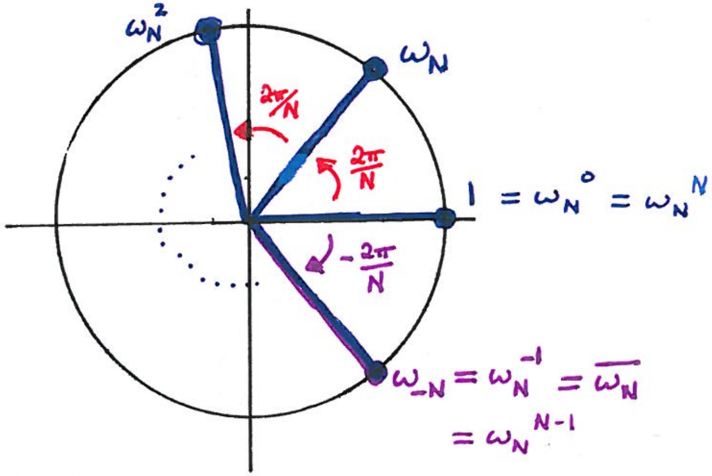
- $\omega_N^N = \omega_N^0 = 1$
- $\omega_N^{N+1} = \omega_N$
- $\omega_N^{N+2} = \omega_N^2$

- $\omega_N^{N-1} = \omega_N^{-1}$

Also $\omega_N^{-1} = \overline{\omega_N}$ $\mathbb{C}$-conjugate.

Note: $e^{\left(-\frac{2\pi i}{N}\right)} = \omega_{-N}$ is like ω_N:

→ Divide circle into N pieces then go <u>clockwise</u> instead of <u>counter-clockwise</u>.



Discrete Fourier Transform: $\mathcal{F}\{\underline{f}\} = \underline{c}$

is given by

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} f_n \left(\overline{\omega_N}^k\right)^n$$

"Winds f around the circle <u>clockwise</u>, with steps of size <u>$1/N$</u> of circle, moving <u>k</u> steps at a time."

Inverse Fourier Transform: $\mathcal{F}^{-1}\{\underline{c}\} = \underline{f}$

$$f_n = \sum_{k=0}^{N-1} c_k \left(\omega_N^n\right)^k$$

Note: Fourier Transform "winds" the function $\underline{f}$ clockwise around circle

Inverse Fourier "unwinds" the coefficients $\underline{c}$ counter-clockwise to get back to $\underline{f}$.

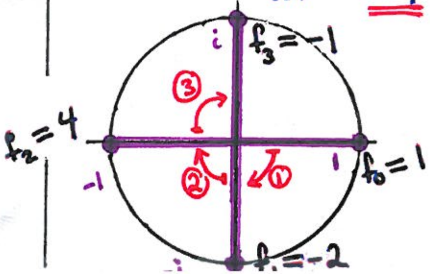
[To save space, in the following examples, we will write our vectors horizontally:
$$\underline{f} = (f_0, f_1, f_2, f_3, \dots, f_{N-1})$$]

EX: Compute $\mathcal{F}\{\underline{f}\}$ for $\underline{f} = (1, -2, 4, -1)$ — $1 = f_0$, $-2 = f_1$, $4 = f_2$, $-1 = f_3$, length $= 4$

c_0 [Note: $\omega_N^0 = 1$ so
$$c_0 = \frac{1}{N}\sum f_n(\bar{\omega}_N^0)^n = \frac{1}{N}\sum f_N \cdot 1$$
= average value of $\underline{f}$]

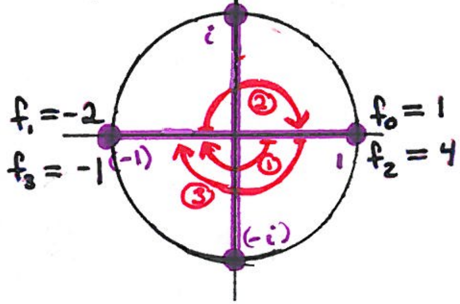
$$c_0 = \tfrac{1}{4}(1-2+4-1) = \tfrac{1}{2}$$

c_1 $\mathcal{F}_1\{\underline{f}\}$ puts $\underline{f}$ on a circle clockwise at $1/4$ turns (because $\underline{f}$ is length 4)



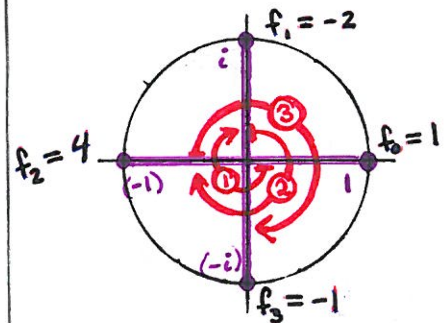
$$c_1 = \tfrac{1}{4}\big(1\cdot 1 + (-2)(-i) + 4(-1) + (-1)i\big) = \tfrac{1}{4}(-3+i)$$

c_2 $\mathcal{F}_2\{\underline{f}\}$ puts $\underline{f}$ on the same circle, taking two steps at a time.



$$c_2 = \tfrac{1}{4}\big(1\cdot 1 + (-2)(-1) + 4\cdot 1 + (-1)(-1)\big) = \tfrac{1}{4}(8) = 2$$

c_3 $\mathcal{F}_3\{\underline{f}\}$ puts $\underline{f}$ on the same circle, taking three steps at a time.



$$c_3 = \tfrac{1}{4}\big(1\cdot 1 + (-2)i + 4(-1) + (-1)(-i)\big) = \tfrac{1}{4}(-3-i)$$

$$\boxed{\underline{c} = \left(\tfrac{1}{2},\ \tfrac{1}{4}(-3+i),\ 2,\ \tfrac{1}{4}(-3-i)\right)}$$

Important simplifications to note:

- c_0 = average value of $\underline{f}$
- $c_{N-k} = \overline{c_k}$ ← $\mathbb{C}$-conjugate

→ We only need to compute half of the c_k because the others are conjugate

$$\underline{c} = (\underset{\text{Real}}{c_0}, c_1, c_2, \dots, \underset{\overline{c_2}}{c_{N-2}}, \underset{\overline{c_1}}{c_{N-1}})$$

EX: Compute $\mathcal{F}\{\underline{f}\}$ for

$$\underline{f} = \underbrace{(\underset{f_0}{2}, \underset{f_1}{-1}, \underset{f_2}{1}, \underset{f_3}{-2}, \underset{f_4}{3}, \underset{f_5}{0})}_{\text{length}=6}$$

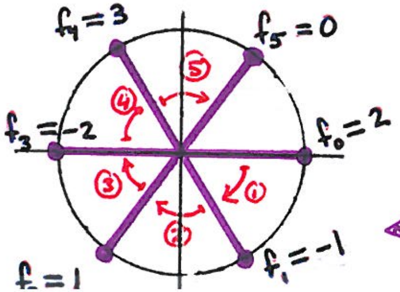
- $\underline{c} = (\underset{\text{Real}}{c_0}, c_1, c_2, \underset{\overline{c_3}}{c_3}, \underset{\overline{c_2}}{c_4}, \underset{\overline{c_1}}{c_5})$

→ Compute c_0, c_1, c_2, c_3:

- $\underline{c_0}$ = average value of f

$$= \tfrac{1}{6}(2 - 1 + 1 - 2 + 3 + 0) = \tfrac{1}{2}$$

- $\underline{c_1}$

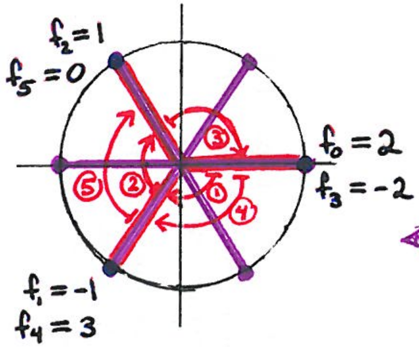


$$\overline{\omega_6} = \frac{1-\sqrt{3}\,i}{2}$$

$$c_1 = \tfrac{1}{6}\Big(2(1) + (-1)\tfrac{1-\sqrt{3}\,i}{2} + (1)\tfrac{-1-\sqrt{3}\,i}{2} + (-2)(-1) + 3\,\tfrac{-1+\sqrt{3}\,i}{2} + 0\,\tfrac{1+\sqrt{3}\,i}{2}\Big)$$

$$= \tfrac{1}{6}\left(\frac{-3+3\sqrt{3}\,i}{2}\right) = \tfrac{1}{4}(-1+\sqrt{3}\,i)$$

$\underline{c_2}$

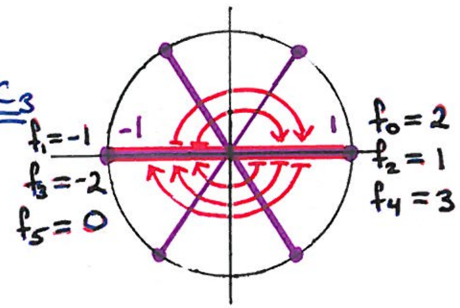


$$\overline{\omega_6}^{\,2} = \frac{-1-\sqrt{3}\,i}{2}$$

$$c_2 = \tfrac{1}{6}\Big(2\cdot 1 + (-1)\tfrac{-1-\sqrt{3}\,i}{2} + 1\cdot\tfrac{-1+\sqrt{3}\,i}{2} + (-2)\cdot 1 + 3\cdot\tfrac{-1-\sqrt{3}\,i}{2} + 0\cdot\tfrac{-1+\sqrt{3}\,i}{2}\Big)$$

$$= \tfrac{1}{6}\left(-\tfrac{3}{2} - \tfrac{\sqrt{3}}{2}\,i\right) = \tfrac{1}{12}(-3-\sqrt{3}\,i)$$

$\underline{c_3}$



$$c_3 = \tfrac{1}{6}\Big(2 - (-1) + 1 - (-2) + 3 - 0\Big) = \tfrac{1}{6}\cdot 9 = \tfrac{3}{2}$$

$$\boxed{\underline{c} = \Big(\tfrac{1}{2},\ \tfrac{1}{4}(-1+\sqrt{3}\,i),\ \tfrac{1}{12}(-3-\sqrt{3}\,i),\ \tfrac{3}{2},\ \tfrac{1}{12}(-3+\sqrt{3}\,i),\ \tfrac{1}{4}(-1-\sqrt{3}\,i)\Big)}$$

Normally I will not redraw the whole ω_n^n circle for each calculation...

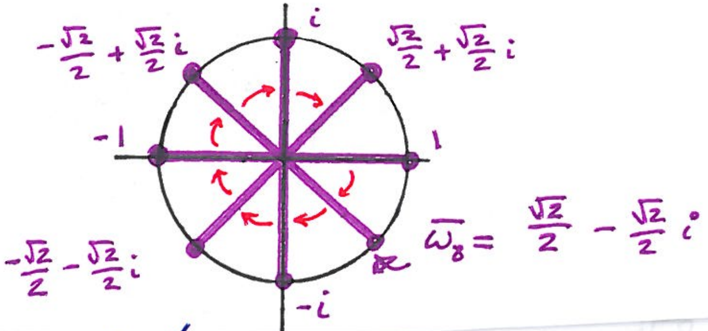
Note: This vector is periodic with freq = 2 (so most $c_k = 0$)

EX: Compute all Fourier coefficients of

$$\underline{f} = [\,-1 \quad 0 \quad 1 \quad 0 \quad -1 \quad 0 \quad 1 \quad 0\,]$$

$$= f_0 \quad f_1 \quad f_2 \quad f_3 \quad f_4 \quad f_5 \quad f_6 \quad f_7$$

length = 8



$$c_0 = \tfrac{1}{8}\left(-1 + 0 + 1 + 0 + (-1) + 0 + 1 + 0\right) = 0$$

$$c_1 = \tfrac{1}{8}\left(-1\cdot 1 + 0\left(\tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2}i\right) + 1\cdot(-i) + 0\left(-\tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2}i\right) + (-1)(-1) + 0\left(-\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2}i\right) + 1(i) + 0\left(\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2}i\right)\right)$$

$$= 0$$

$$c_2 = \tfrac{1}{8}\left(-1\cdot 1 + 0\cdot(-i) + 1\,(-1) + 0\,(i) + (-1)\cdot 1 + 0\,(-i) + 1\,(-1) + 0\,(i)\right)$$

(take every other coeff) (loop around & repeat)

$$= \tfrac{1}{8}(-4) = -\tfrac{1}{2}$$

$$c_3 = \tfrac{1}{8}\left(-1\cdot 1 + 0\left(-\tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2}i\right) + 1\,(i) + 0\left(\tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2}i\right) + (-1)(-1) + 0\left(\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2}i\right) + 1(-i) + 0\left(-\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2}i\right)\right)$$

(take every third coeff) loop around & continue

$$= 0$$

$$c_4 = \tfrac{1}{8}\left(-1\cdot 1 + 0\,(-1) + 1\cdot 1 + 0\,(-1) + (-1)(1) + 0\,(-1) + 1\,(1) + 0\,(-1)\right)$$

(take every 4^{th} coefficient)

$$= 0$$

$$c_5 = c_{8-3} = \overline{c_3} = 0 \qquad c_6 = c_{8-2} = \overline{c_2} = -\tfrac{1}{2} \qquad c_7 = c_{8-1} = \overline{c_1} = 0$$